TWO-DIMENSIONAL TRANSONIC GAS FLOW FAR FROM A PROFILE LOCATED IN THE CHANNEL

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In this paper the characteristic solution of the two-dimensional transonic flow of a gas is constructed far from a symmetrical profile placed at zero angle of attack along the axis of a channel with parallel walls. The indicated characteristic solution of the stream function $\psi(\theta, \eta)$, which satisfies the Tricomi equation, is constructed in the plane θ, η by the method of singular integral equations. It is shown that for unlimited widening of the channel, preservation of the choking condition and fulfillment of a certain other condition, this characteristic solution transforms into the characteristic solution of two-dimensional free sonic flow, found by Frankl' [1] and Guderley.

1. We shall examine the flow over a thin symmetrical convex profile placed at zero angle of attack along the axis of a channel with parallel walls of width $2L$, by a two-dimensional transonic gas flow at the choking condition when a sonic line AB (Fig.1) arises between the body and the walls.

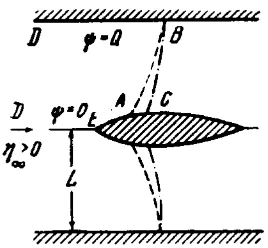

Fig. 1

Let the stream function $\psi(\theta, \eta)$ satisfy the Tricomi equation

$$\eta\psi_{\theta\theta} + \psi_{\eta\eta} = 0 \qquad (1.1)$$

and assume the value $\psi = 0$ on the axis DE and on the boundary of the body; on the wall DB of the channel $\psi = Q$. Here θ is the angle between the velocity vector and the axis of the channel, η is a known function of the velocity modulus and $2Q$ is the gas flow in the channel.

Far in front of the profile the flow closely resembles a homogeneous subsonic flow ($\eta_\infty > 0$), behind the sonic line AB the flow becomes supersonic. Because of the symmetry of the picture we limit ourselves to a study of the flow above the axis DE to the limiting characteristic BC. The flow region $DEACBD$ transforms in the plane $\theta\eta$ into a certain region (Fig.2). The stagnation point of flow E transforms to infinity in the plane $\theta\eta$. Curve CAE in Fig.2, which corresponds to the boundary of the body, is usually not known. For sufficiently great width $2L$ of the channel, the influence of shape of the profile on the subsonic flow of the gas far from the body is insignificant. The region of gas flow far from the profile (where $\theta \approx 0$) will be reflected into the vicinity of axis η (BDE) in Fig.2; function $\psi(\theta, \eta)$ for $\theta \approx 0$ will depend weakly on the form of curve EAC for constant Q and η_∞. The purpose of the following will be to obtain the function $\psi(\theta, \eta)$ which describes the transonic flow far from the body in the channel (for $\theta \approx 0$) where the shape of the body has little influence on the gas flow.

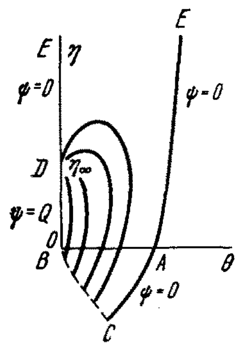

Fig. 2

In order to have a shock-free continuation of flow beyond the sonic line AB, it is sufficient for the stream function $\psi(\theta, \eta)$ in the vicinity of the center of the nozzle B to have the following behavior [3]:

$$\tau(\theta) = Q - 3^{2/3}h\theta^{1/3} + O(\theta), \qquad \nu(\theta) = h\theta^{-1/3} + O(\theta^{1/3}) \quad (\theta \to 0) \tag{1.2}$$

Here

$$\psi(\theta, 0) = \tau(\theta), \qquad \partial\psi(\theta, 0)/\partial\eta = \nu(\theta) \quad (h = \text{const}) \tag{1.3}$$

The desired function $\psi(\theta, \eta)$ must describe the homogeneous flow far ahead of the profile and also the flow in the vicinity of the center of the nozzle B near the wall of the channel DB, where the conditions (1.2) and (1.3) hold.

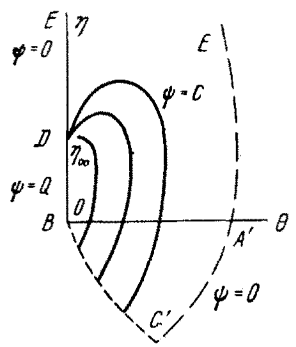

Fig. 3

2. In order to construct a characteristic solution of the flow far from the profile in the channel — a solution, which is independent of the shape of the surface of the body — we proceed in the following way. The curve EAC in Fig.2 is removed to infinity in the $\theta\eta$ plane, retaining the condition $\psi = 0$ on EAC. All other points and boundary conditions for the function $\psi(\theta, \eta)$ will be unchanged. Then we obtain the following boundary value problem for $\psi(\theta, \eta)$: within the region $EDBC'A'E$ (Fig.3) where $A'C'$ is a characteristic of Equation (1.1) moved out to infinity, it is necessary to determine a bounded solution of Equation (1.1) satisfying the following conditions

$$\psi(0, \eta) = \begin{cases} Q, & 0 \leqslant \eta < \eta_\infty \\ 0, & \eta_\infty < \eta < \infty \end{cases} \tag{2.1}$$

$$\psi(\theta, 0) = Q + O(\theta^{1/3}), \qquad \psi_\eta(\theta, 0) = O(\theta^{-1/3}) \quad (\theta \to 0) \tag{2.2}$$

$$\psi = 0 \text{ at infinity } EA' \tag{2.3}$$

$$\psi = 0 \text{ on the infinite characteristic } A'C' \tag{2.4}$$

By constructing the solution $\psi(\theta, \eta)$ of this boundary value problem of Tricomi, for $\theta \approx 0$ the desired characteristic solution of the flow is obtained far from the profile in the channel.

Following Gellerstedt [4] the boundary value problem is solved by the method of singular equations in closed form. The solution $\psi(\theta, \eta)$ for $\eta \geqslant 0$ is sought in the form of a sum of solutions of Equation (1.1)

$$\psi(\theta, \eta) = \psi_1(\theta, \eta) + \psi_2(\theta, \eta) \tag{2.5}$$

$$\psi_1(\theta, \eta) = Qt_\infty^{2/3} t^{1/3} \int_0^\infty e^{-\lambda\theta} J_{-1/3}(\lambda t) J_{2/3}(\lambda t_\infty)\, d\lambda, \qquad t = \frac{2}{3}\eta^{3/2}, \quad t_\infty = \frac{2}{3}\eta_\infty^{3/2} \tag{2.6}$$

$$\psi_2(\theta, \eta) = -\gamma \int_0^\infty \nu(s) \left\{ \left[(\theta - s)^2 + \frac{4}{9}\eta^3\right]^{-1/6} - \left[(\theta + s)^2 + \frac{4}{9}\eta^3\right]^{-1/6} \right\} ds$$

$$\gamma = \frac{3^{2/3}\Gamma^3(1/3)}{4\pi^2} \tag{2.7}$$

Here $\nu(\theta) = \psi'_\eta(\theta, 0),\ 0 < \theta < \infty$ is a function subject to determination. It satisfies the conditions: $\nu(\theta) = O(\theta^{-1/3})$ for $\theta \to 0$; $\nu(\theta) \to 0$ for $\theta \to \infty$, which follows from (2.2) and (2.3). It is not difficult to verify that function $\psi(\theta, \eta)$ of Equation (2.5) satisfies conditions (2.1) and (2.3). Condition (2.4) on characteristic $C'A'$ is equivalent to the integral relationship ([5], p.381)

$$\tau(\theta) = \gamma \int_\theta^\infty \frac{\nu(s)\, ds}{(s - \theta)^{1/3}} \qquad (0 < \theta < \infty) \tag{2.8}$$

where the notation of (1.3) is used; the constant γ is defined in (2.7).

The second integral relationship between functions $\tau(\theta)$ and $\nu(\theta)$ is obtained from Equations (2.5) to (2.7)

$$\tau(\theta) = \tau_1(\theta) - \gamma \int_0^\infty \nu(s) \{|\theta - s|^{-1/3} - (\theta + s)^{-1/3}\}\, ds \quad (0 < \theta < \infty) \tag{2.9}$$

Here $\tau_1(\theta) = \psi_1(\theta, 0)$. Eliminating the quantity $\tau(\theta)$ from (2.8) and (2.9), we shall obtain an integral equation for the determination of function $\nu(\theta)$. After some transformations [5], this integral equation will take the form

$$\nu(\theta) - \frac{1}{\pi\sqrt{3}} \int_0^\infty \left(\frac{1}{s - \theta} + \frac{1}{s + \theta}\right) \nu(s)\, ds = F(\theta) \quad (0 < \theta < \infty) \tag{2.10}$$

Here

$$F(\theta) = -\frac{1}{\sqrt{3}\,\pi\gamma} \int_\theta^\infty \frac{\tau_1'(s)\, ds}{(s - \theta)^{2/3}}, \qquad \tau_1'(\theta) = \frac{d}{d\theta}\psi_1(\theta, 0) \tag{2.11}$$

The singular integral equation (2.10) has an automorphic kernel. By substitution of $\theta^2 = x$ and $s^2 = y$ the equation readily reduces to the characteristic integral equation. Solution of Equation (2.10) is written in closed form ([6] Section 47). Since the desired function $\nu(\theta)$ satisfies conditions enumerated after Equation (2.7), then in (2.10) the index of the equation will be $k = 0$; therefore in the indicated class, Equation (2.10) has a unique solution, if function $F(\theta)$ satisfies certain conditions of smoothness ([6], Section 47).

The canonic function of Equation (2.10) in the given class is $Z(\theta) = \theta^{-1/3}$, and the solution has the form

$$\nu(\theta) = \frac{3}{4}\left[F(\theta) + \frac{1}{\pi\sqrt{3}}\int_0^\infty \left(\frac{s}{\theta}\right)^{1/3}\left(\frac{1}{s-\theta} + \frac{1}{s+\theta}\right)F(s)\,ds\right] = R[F(\theta)]$$

$$(0 < \theta < \infty) \tag{2.12}$$

The function $\nu(\theta)$ which was found, determines the solution $\psi(\theta, \eta)$ of the problem of Tricomi. It remains to compute the integrals encountered and to verify the fulfillment of condition (2.2). Functions $\tau_1(\theta) = \psi_1(\theta, 0)$ and $\tau_1'(\theta) = d\psi_1(\theta, 0)/d\theta$ are determined from (2.6)

$$\tau_1(\theta) = Qt_\infty^{2/3}\frac{2^{1/3}}{\Gamma(2/3)}\int_0^\infty \lambda^{-1/3}e^{-\lambda\theta}J_{1/3}(t_\infty\lambda)\,d\lambda = \frac{Q}{2}\left(\frac{3}{2}\right)^{1/3}\gamma\left(\frac{t_\infty^2}{t_\infty^2+\theta^2}\right)^{1/3}\times$$

$$\times F\left(\frac{2}{3}, \frac{1}{2}; \frac{5}{3}; \frac{t_\infty^2}{t_\infty^2+\theta^2}\right) \tag{2.13}$$

$$\tau_1'(\theta) = -Qt_\infty^{2/3}\frac{2^{1/3}}{\Gamma(2/3)}\int_0^\infty \lambda^{2/3}e^{-\lambda\theta}J_{1/3}(t_\infty\lambda)\,d\lambda = -Q\left(\frac{2}{3}\right)^{1/3}\gamma\frac{t_\infty^{4/3}}{(\theta^2+t_\infty^2)^{7/6}}$$

The hypergeometric series in (2.13) converges absolutely for all real values of the parameters.

For $\theta \to 0$ we obtain $\tau_1(\theta) = Q + O(\theta)$. Substituting the integral for $\tau_1'(\theta)$ from (2.13) into (2.11) and changing the order of integration we find

$$F(\theta) = \frac{Qt_\infty^{2/3}2^{1/3}\Gamma(1/3)}{\sqrt{3}\pi\gamma\Gamma(2/3)}\int_0^\infty \lambda^{1/3}e^{-\lambda\theta}J_{1/3}(t_\infty\lambda)\,d\lambda =$$

$$= \frac{3^{5/6}}{\pi 2^{4/3}}\frac{Q}{t_\infty^{2/3}}\frac{t_\infty^2}{\theta^2+t_\infty^2}F\left(1, \frac{1}{6}; \frac{5}{3}; \frac{t_\infty^2}{\theta^2+t_\infty^2}\right) \tag{2.14}$$

where the hypergeometric series converges absolutely for all real values of the parameters. Substituting the series for $F(\theta)$

$$F(\theta) = \frac{3^{5/6}Q}{\pi 2^{4/3}t_\infty^{2/3}}\sum_{n=0}^\infty \frac{\Gamma(n+1/6)\Gamma(5/3)}{\Gamma(1/6)\Gamma(n+5/3)}t_\infty^{2n+2}(\theta^2+t_\infty^2)^{-n-1} \tag{2.15}$$

into Equation (2.12) we find $\nu(\theta)$. It is necessary here to compute the function $R[(\theta^2+t_\infty^2)^{-n-1}]$ $(n = 0, 1, 2 \ldots)$, where the linear operator R is determined in (2.12). After a slight transformation we obtain

$$R[(\theta^2+t_\infty^2)^{-n-1}] = \frac{3}{4}\left[(\theta^2+t_\infty^2)^{-n-1} + \frac{1}{\pi\sqrt{3}}\theta^{-1/3}\int_0^\infty \frac{s^{1/6}(s+t_\infty^2)^{-n-1}}{s-\theta^2}\,ds\right] \tag{2.16}$$

The singular integral entering into this is computed as follows (*):

*) See footnote on the next page.

$$\int_0^\infty \frac{s^{1/6}(s+t_\infty^2)^{-n-1}}{s-\theta^2}\,ds = -\pi\sqrt{3}\,\theta^{1/3}(\theta^2+t_\infty^2)^{-n-1} -$$

$$-\frac{t_\infty^{1/3-2n}}{(t_\infty^2+\theta^2)}B\left(n-\frac{1}{6},\frac{7}{6}\right)F\left(-n,1;-n+\frac{7}{6};\frac{t_\infty^2}{\theta^2+t_\infty^2}\right) \quad (n=0,1,2\ldots)$$

Then in (2.16) some terms disappear and from (2.15) and (2.12) we determine the function $\nu(\theta)$

$$\nu(\theta) = \theta^{-1/3}\frac{3(3/2)^{1/3}}{4\pi}\frac{Qt_\infty^{5/3}}{t_\infty^2+\theta^2}\sum_{n=0}^{\infty}\frac{\Gamma(n+1/6)\,\Gamma(n-1/6)\,\Gamma(5/3)}{\Gamma(1/6)\,\Gamma(-1/6)\,\Gamma(n+5/3)\,\Gamma(n+1)}\times$$

$$\times F\left(-n,1;-n+\frac{7}{6};\frac{t_\infty^2}{t_\infty^2+\theta^2}\right) \qquad (2.17)$$

Here equations for the transformation of gamma functions are utilized

$$\Gamma(1+z)=z\Gamma(z), \qquad \Gamma(1-z)\Gamma(z)=\frac{\pi}{\sin\pi z}$$

Series (2.17) converges absolutely. This follows from representation of $\nu(\theta)$ in the form (2.12) and (2.14). Therefore, regrouping the terms in series (2.17) in order of increasing powers of the argument of polynomials, we obtain

$$\nu(\theta)=\theta^{-1/3}\frac{Qt_\infty^{5/3}}{t_\infty^2+\theta^2}\frac{4(2)^{1/3}}{5\pi(3)^{1/6}}F\left(\frac{1}{6},1;\frac{11}{6};\frac{t_\infty^2}{t_\infty^2+\theta^2}\right) \qquad (2.18)$$

Differentiating (2.9) with respect to θ and eliminating $\nu(\theta)$ with the aid of (2.8), we arrive [5] at a singular equation for the derivative $\tau'(\theta)$.

$$\tau'(\theta)-\frac{1}{\pi\sqrt{3}}\int_0^\infty\left(\frac{s}{\theta}\right)^{1/3}\left(\frac{1}{s-\theta}+\frac{1}{s+\theta}\right)\tau'(s)\,ds=\frac{2}{3}\tau_1'(\theta) \qquad (0<\theta<\infty)$$

With consideration of (1.2),(2.3),(2.10),(2.12) and (2.13) its solution has the form

$$\tau'(\theta)=-\theta^{-2/3}\frac{Qt_\infty^{5/3}}{\theta^2+t_\infty^2}\frac{2^{4/3}}{\sqrt{3}\pi}F\left(-\frac{1}{6},1;\frac{7}{6};\frac{t_\infty^2}{t_\infty^2+\theta^2}\right)$$

Condition (1.2) is satisfied here.

Thus the solution for the Tricomi problem (2.1) to (2.4) is constructed. The solution is determined by Equations (2.5) to (2.7) and (2.18). Integrals in (2.6) and (2.7) can be presented in the form of series. For $\theta \approx 0$ solution $\psi(\theta,\eta)$ describes transonic two-dimensional flow of a gas far from the profile placed in a channel with parallel walls.

In order to satisfy the condition $\psi = 0$ on the boundary of the profile, it is necessary to add to the constructed solution $\psi(\theta,\eta)$ a regular solution ψ_3 of Equation (1.1) which vanishes for $\theta = 0$, $0 \leqslant \eta < \infty$ such that the sum $\psi + \psi_3$ is equal to zero along the curve *EAC* in Fig.2.

3. It will be shown that in case of indefinite widening of the channel,

*) In Equation 3.228.5 from [7] for $c>0$ the factor π is left out in front of all terms (see for example equation 3.222.2 from [7] for $a<0$ and primary source).

while maintaining the chocking condition, and in satisfying a certain other condition, the constructed solution transforms into the characteristic solution of two-dimensional free sonic flow, found by Frankl' [1 and 2] and Guderley

$$\psi_0(\theta, \eta) = C\rho^{-5/3}[(1-s)^{1/3}(1/3+s) - (1+s)^{1/3}(1/3-s)]$$

$$\rho = \sqrt{\theta^2 + 4/9\,\eta^3}, \quad s = \theta/\rho \tag{3.1}$$

here C is some constant which depends on the size of the profile. This self-similar solution of Equation (1.1) becomes zero for $\theta = 0$, $0 < \eta < \infty$, it is regular on the limiting characteristic BC' and on BA' (Fig.3), it turns to infinity at the point B $(\theta = \eta = 0)$ and tends to zero at infinity of the plane θ, η for $\rho \geqslant 0$.

$$\tag{3.2}$$

Designating

$$\tau_0(\theta) = \psi_0(\theta, 0) = C\cdot\frac{2^{4/3}}{3}\theta^{-5/3}, \qquad \nu_0(\theta) = \frac{\partial\psi_0(\theta,0)}{\partial\eta} = C\frac{2^{7/3}}{3^{5/6}}\theta^{-7/3} \quad (0<\theta<\infty)$$

we readily find the relationship between $\tau_0(\theta)$ and $\nu_0(\theta)$

$$\tau_0(\theta) = \gamma\int_\theta^\infty \frac{\nu_0(s)\,ds}{(s-\theta)^{1/3}} \qquad (0<\theta<\infty) \tag{3.3}$$

Here the constant γ is determined in (2.7). Relationship (3.3) shows that $\psi_0(\theta, \eta)$ vanishes on the characteristic $A'C'$ (Fig.3).

We shall examine the limit of solution $\psi(\theta, \eta)$ of (2.5) for unlimited widening of the channel $(L \to \infty)$ while the choking condition is maintained $(Q \to \infty$, $\eta_\infty \to 0)$. Then point D in Fig.3 approaches point B and the value of $\psi(\theta, \eta)$ increases without bound on section DB . The limit of the sum of the series from (2.17) for $t_\infty = 2/3\,\eta_\infty^{3/2} \to 0$ is equal to the constant

$$F\left(\frac{1}{6}, -\frac{1}{6}; \frac{5}{3}; 1\right) = \frac{32}{15\cdot 2^{1/3}\sqrt{3}}$$

Therefore the limit of function $\nu(\theta)$ for $t_\infty \to 0$, $Q \to \infty$ is determined by the limit of the product $Q \times t_\infty^{5/3}$ from (2.17) which can be equal to zero, to infinity or to a finite quantity, if it is not assumed that the body dimensions are preserved when the channel is widened. It will be shown that for the condition

$$\lim Qt_\infty^{5/3} = A > 0 \quad \text{for} \quad t_\infty \to 0 \qquad (A = \text{const}) \tag{3.4}$$

the constructed characteristic solution $\psi(\theta, \eta)$ transforms into the characteristic solution (3.1) for free sonic flow near the body. Denoting the limit of function $\psi(\theta, \eta)$, when condition (3.4) is satisfied, by $\psi_*(\theta, \eta)$ and the limit of function $\nu(\theta)$ by $\nu_*(\theta)$, we obtain from (2.17)

$$\nu_*(\theta) = \frac{4}{5\pi}\frac{2^{1/3}}{3^{1/6}}A\theta^{-7/3} \qquad (0<\theta<\infty) \tag{3.5}$$

i.e. function $\nu_*(\theta)$ can differ from function $\nu_0(\theta)$ in (3.2) only by a constant factor. By equating these two we obtain

$$C = \frac{3\sqrt{3}}{5\pi}A \tag{3.6}$$

If $\tau_*(\theta)$ is the limit of $\tau(\theta)$ for $t_\infty \to 0$ and condition (3.4), then condition (2.8) for $\psi_*(\theta, \eta)$ is rewritten in the form

$$\tau_*(\theta) = \gamma \int_\theta^\infty \frac{\nu_*(s)\,ds}{(s-\theta)^{1/3}} \qquad (0<\theta<\infty) \tag{3.7}$$

Then from equality $\nu_*(\theta) \equiv \nu_0(\theta),\ 0<\theta<\infty$ and relationships (3.7) and (3.3) it follows that $\tau_*(\theta) \equiv \tau_0(\theta), 0<\theta<\infty$. Since

$$\psi_*(0, \eta) = \psi_0(0, \eta) = 0, \quad 0<\eta<\infty,$$

then because of uniqueness of solution of the Dirichlet problem for $\eta \geqslant 0$ and the Cauchy problem for $\eta \leqslant 0$, we obtain $\psi_*(\theta, \eta) \equiv \psi_0(\theta, \eta)$ in the region of Fig.3.

Condition (3.4) can be written in a different form. For $L \approx \infty$ and $\eta_\infty \approx 0$ we have

$$Q = O(L), \quad t_\infty = {}^2/_3\,\eta_\infty{}^{3/2} = O\,[(1-M_\infty)^{3/2}]$$

where M_∞ is the choking Mach number of uniform flow far ahead of the profile in the channel. Therefore we obtain from (3.4)

$$1 - M_\infty = B\left(\frac{l}{L}\right)^{2/3}, \qquad M_\infty \approx 1 \tag{3.8}$$

where l is a characteristic linear dimension of the profile, B is a non-dimensional constant which depends on the shape of the profile.

It remains to clarify the physical meaning of condition (3.8) for $L \to \infty$ and $M_\infty \to 1$. We shall demonstrate that condition (3.8) is equivalent to the requirement of maintaining the dimensions of the profile when the channel is widenend at least for certain shapes of profiles. For this it is necessary to solve the boundary value problem in Fig.2 for a definite profile, to express the characteristic dimension l of the profile through Q, η_∞ and the shape of the profile and to check condition (3.8) for $M_\infty \to 1$ and for constant l. The problem of a wedge in the channel was solved analytically by Marschner [8] by means of characteristic solutions of Equation (1.1) introduced by Guderley for the condition $t_\infty < \theta_0$, where θ_0 is the half-angle of the wedge. For the condition of constant wedge dimension the equation for the choking Mach number M_∞ has the form (3.8).

The problem of a wedge in a channel under choking conditions was also solved by Morioka [9] by the relaxation method. Calculations were made for three values of the parameter l/L where l is the length of the wedge for fixed value of angle θ_0. For $M \approx 1$ the equation for choking Mach number M_∞ of the form (3.8) which was obtained by Marschner agrees satisfactorily with numerical values of [9]. An equation of the form (3.8) was also obtained by Guderley [10 and 11] in the solution of the problem of gas flow near a flat plate placed at an angle of attack in a channel with parallel walls. It may be assumed that condition (3.8)will be fulfilled for some convex bodies when the channel is widened, when the choking condition of the channel is maintained and when the dimensions of the profile are preserved. Then the two-dimensional transonic flow near the profile placed into a channel with parallel walls will transform into the free sonic flow near the profile.

We note that recently the characteristic solution of free sonic flow near a body was checked experimentally [12].

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